Multi-normed spaces

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The lectures are based on a manuscript : 'Multi-normed spaces' by H. G. Dales and M. E. Polyakov [**2**], which is in preparation. Some topics of [**2**] are not mentioned here. For a version of the manuscript or comments on the topic, please contact

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Full acknowledgements and references are given in this manuscript.

There were also two lectures in Bangalore on 'Multi-Banach algebras and amenable groups', by Paul Ramsden. It is hoped that this work will be the basis of a future manuscript [**3**]; see [**6**]. Some motivation for the existence of a theory of multi-norms was given in the lectures of Ramsden.

1. Lecture - The axioms

1.1. Some background. The theory of multi-norms given here assumes a background in normal functional analysis, at the level of a masters course; we mention here some notation that we shall use (see also [**1**]) and a few extra points that may not be covered in such a course.

Linear spaces are always supposed to be over the complex field $\mathbb{C}$, unless stated otherwise; however an analogous theory for spaces over $\mathbb{R}$ is also given in [**2**]. The real space underlying a linear space E is denoted by $E_{\mathbb{R}}$.

The closed unit ball of a normed space E is denoted by $E_{[1]}$.

Let E and F be normed spaces, and let $T \in \mathcal{B}(E, F)$, the space of bounded linear operators from E to F. Then the *dual* T' of T is defined by the equation

$$\langle x, T'\lambda\rangle = \langle Tx, \lambda\rangle \quad (x \in E, \lambda \in F')\,;$$

we have $T' \in \mathcal{B}(F', E')$ and $\|T'\| = \|T\|$. The space E is *linearly homeomorphic to* F if there is a bijection $T \in \mathcal{B}(E, F)$ with $T^{-1} \in \mathcal{B}(F, E)$; such a map T is a *linear homeomorphism* or an *isomorphism*. In this case, the *Banach–Mazur distance* from E to F is

$$d(E, F) = \inf\{\|T\| \left\|T^{-1}\right\| : T \in \mathcal{B}(E, F) \text{ is an isomorphism}\}.$$

The projective tensor product of E and F is denoted by $(E \otimes F, \pi)$; its completion is $(E \widehat{\otimes} F, \pi)$.

Several proofs implicitly use Hölder's inequality in the following form. Take $p > 1$, and let q be the conjugate index to p. Then, for each $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \le \left(\int_\Omega |f|^p\right)^{1/p} \left(\int_\Omega |g|^q\right)^{1/q} .$$

We shall refer to the standard Banach sequence spaces $\ell^p = \ell^p(\mathbb{N})$ for $p \in [1, \infty]$ and the space c_0 of null sequences; we write δ_i for the sequence $(\delta_{i,j} : j \in \mathbb{N})$.

Let $m, n \in \mathbb{N}$. Then we can identify $\mathbb{M}_{m,n}$ with the Banach space $\mathcal{B}(\ell^\infty_n, \ell^\infty_m)$, so that $(\mathbb{M}_{m,n}, \|\cdot\|)$ is a Banach space. Indeed, the formula for the norm in $\mathbb{M}_{m,n}$ of an element $a = (a_{ij})$ is then

$$\|a\| = \|a : \ell^\infty_n \to \ell^\infty_m\| = \max\left\{\sum_{j=1}^n |a_{ij}| : i = 1, \dots, m\right\} . \tag{1.1}$$

In the case where $m = n$, we obtain a unital Banach algebra $(\mathbb{M}_n, \|\cdot\|)$. More generally, let $p, q \in [1, \infty]$. Then we can also identify $\mathbb{M}_{m,n}$ with

$\mathcal{B}(\ell_n^p, \ell_m^q)$, and denote the norm of $a \in \mathbb{M}_{m,n}$ by $\|a : \ell_n^p \to \ell_m^q\|$. For example,

$$\left\|a : \ell_n^1 \to \ell_m^1\right\| = \max\left\{\sum_{i=1}^m |a_{ij}| : j = 1, \dots, n\right\}. \tag{1.2}$$

The final result is the *principle of local reflexivity.*

PROPOSITION 1.1. *Let E be a Banach space, let X be a finite-dimensional subspace of E'', let F be a finite subset of E', and take $\varepsilon > 0$. Then there is an injective linear map $S : X \to E$ with $S \mid X \cap E$ the identity on $X \cap E$, with $\|S\| \|S^{-1} : S(X) \to X\| < 1 + \varepsilon$, and with*

$$\langle S(\Lambda),\, \lambda\rangle = \langle \Lambda,\, \lambda\rangle \quad (\lambda \in F,\, \Lambda \in X)\,.$$

□

1.2. The axioms. We begin with our definition of a multi-norm. Here $\mathfrak{S}_n$ is the symmetric group on n symbols.

DEFINITION 1.2. *Let $(E, \|\cdot\|)$ be a normed space. A* multi-norm *on $\{E^n : n \in \mathbb{N}\}$ is a sequence*

$$(\|\cdot\|_n) = (\|\cdot\|_n : n \in \mathbb{N})$$

such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$ (so that $\|\cdot\|_1$ is the initial norm*), and such that the following Axioms* (A1)–(A4) *are satisfied for each $n \in \mathbb{N}$:*

(A1) *for each $\sigma \in \mathfrak{S}_n$ and $x \in E^n$, we have*

$$\left\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\right\|_n = \|(x_1, \dots, x_n)\|_n \;;$$

(A2) *for each $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $x \in E^n$, we have*

$$\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \leq (\max|\alpha_i|)\, \|(x_1, \dots, x_n)\|_n \;;$$

(A3) *for each $x_1, \dots, x_n \in E$, we have*

$$\|(x_1, \dots, x_n, 0)\|_{n+1} = \|(x_1, \dots, x_n)\|_n \;;$$

(A4) *for each $x_1, \dots, x_n \in E$, we have*

$$\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, x_n)\|_n \,.$$

Now $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.

DEFINITION 1.3. *Let $(E, \|\cdot\|)$ be a normed space. A* dual multi-norm *on $\{E^n : n \in \mathbb{N}\}$ is a sequence*

$$(\|\cdot\|_n) = (\|\cdot\|_n : n \in \mathbb{N})$$

such that $\|\cdot\|_n$ *is a norm on* E^n *for each* $n \in \mathbb{N}$*, such that* $\|x\|_1 = \|x\|$ *for each* $x \in E$*, and such that the Axioms* (A1), (A2), (A3) *and the following modified form of Axiom* (A4) *are satisfied for each* $n \in \mathbb{N}$*:*

(B4) *for each* $x_1, \dots, x_n \in E$*, we have*

$$\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, 2x_n)\|_n \, .$$

Now $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *is a* dual multi-normed space*.*

We use the terms *multi-Banach space* and *dual multi-Banach space* when $(E, \|\cdot\|)$ is complete; this ensures that each $(E^n, \|\cdot\|_n)$ is a Banach space.

1.3. Elementary consequences of the axioms. The following are immediate consequences of the axioms for multi-normed and dual multi-normed spaces. Many more easy consequences are given in [**2**].

Initially, we suppose that $(E, \|\cdot\|)$ is a complex normed space, and that $(\|\cdot\|_n : n \in \mathbb{N})$ is a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that Axioms (A1)–(A3) are satisfied. Thus our first two results apply to both multi-normed spaces and to dual multi-normed spaces.

Lemma 1.4. *Let* $x_1, \dots, x_n \in E$*, and* $\zeta_1, \dots, \zeta_n \in \mathbb{T}$*. Then*

$$\|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n = \|(x_1, \dots, x_n)\|_n \, . \qquad \square$$

Lemma 1.5. *Let* $x_1, \dots, x_n \in E$*. Then*

$$\max \|x_i\| \le \|(x_1, \dots, x_n)\|_n \le \sum_{i=1}^{n} \|x_i\| \le n \max \|x_i\| \, . \qquad \square$$

Proposition 1.6. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, and let* $k \in \mathbb{N}$*. Set* $\zeta_k = \exp(2\pi \mathrm{i}/k)$*. Then*

$$\|(x_1, \dots, x_k)\|_k \le \frac{1}{k} \sum_{j=1}^{k} \left\| \sum_{m=1}^{k} \zeta_k^{jm} x_m \right\| \quad (x_1, \dots, x_k \in E)\, . \qquad \square$$

1.4. Standard constructions.

Proposition 1.7. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space.*

(i) *Let* F *be a linear subspace of* E*. Then* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *is a multi-normed space.*

(ii) *Let* F *be a closed linear subspace of* E*. Then*

$$(((E/F)^n, \|\cdot\|_n) : n \in \mathbb{N})$$

is a multi-normed space, where $\|\cdot\|_n$ *is now defined by*

$$\|(x_1+F,\dots,x_n+F)\|_n=\inf\{\|(y_1,\dots,y_n)\|_n : y_i\in x_i+F \ \ (i\in\mathbb{N}_n)\}$$

for $x_1,\dots,x_n\in E$. □

1.5. Theorems on duality. Let $(E,\|\cdot\|)$ be a normed space, let $n\in\mathbb{N}$, and let $\|\cdot\|_n$ be any norm on the space E^n. The dual norm on the space $(E')^n$ is denoted by $\|\cdot\|_n'$, so that, explicitly, for each $\lambda_1,\dots,\lambda_n\in E'$, the value $\|(\lambda_1,\dots,\lambda_n)\|_n'$ is equal to

$$\sup\left\{\left|\sum_{j=1}^n\langle x_j,\,\lambda_j\rangle\right| : x_1,\dots,x_n\in E,\ \|(x_1,\dots,x_n)\|_n\leq 1\right\}.$$

Now let $((E^n,\|\cdot\|_n):n\in\mathbb{N})$ be a multi-normed space or a dual multi-normed space. Then it follows that $((E^n)',\|\cdot\|_n')$ is linearly homeomorphic to $(E')^n$ (with the product topology from E'). Thus we have defined a sequence $(\|\cdot\|_n':n\in\mathbb{N})$ such that $\|\cdot\|_n'$ is a norm on $(E')^n$ for each $n\in\mathbb{N}$. Clearly $\|\lambda\|_1'=\|\lambda\|'$ for each $\lambda\in E'$.

The following two theorems give the duality relations that we should like.

THEOREM 1.8. *Let* $((E^n,\|\cdot\|_n):n\in\mathbb{N})$ *be a multi-normed space. Then* $(((E')^n,\|\cdot\|_n'):n\in\mathbb{N})$ *is a dual multi-Banach space.* □

THEOREM 1.9. *Let* $((F^n,\|\cdot\|_n):n\in\mathbb{N})$ *be a dual multi-normed space. Then* $(((F')^n,\|\cdot\|_n'):n\in\mathbb{N})$ *is a multi-Banach space.* □

1.6. Reformulation of the axioms. There are different ways of characterizing multi-norms; some may be more attractive and natural than the initial version. We give one reformulation here; another will be given later

Let E be a linear space, and let $m,n\in\mathbb{N}$. Then $\mathbb{M}_{m,n}$ acts as a map from E^n to E^m in the obvious way; in particular, E^n is a left $\mathbb{M}_n$-module. Our reformulation requires these actions to be 'Banach' actions, so that, for each $m,n\in\mathbb{N}$, we have

$$\|a\cdot x\|_m\leq\|a\|\,\|x\|_n \quad (x\in E^n,\,a\in\mathbb{M}_{m,n}),$$

where $\|a\|=\|a:\ell_n^\infty\to\ell_m^\infty\|$ denotes the norm of a as a map from ℓ_n^∞ to ℓ_m^∞. In particular, E^n is a Banach left $\mathbb{M}_n$-module. Let $m,n\in\mathbb{N}$, and let

$$a=(a_{ij})\in\mathbb{M}_{m,n}\,.$$

Then a is a *row-special* matrix if, for each $i=1,\dots,m$, there is at most one non-zero term, say $a_{i,j(i)}$, in the $i^{\text{th.}}$ row.

Theorem 1.10. *Let* $(E, \|\cdot\|)$ *be a normed space, and let*

$$(\|\cdot\|_n : n \in \mathbb{N})$$

be a sequence of norms on the spaces E^n*, respectively, such that* $\|x\|_1 = \|x\| \ \ (x \in E)$*. Then the following are equivalent:*

(a) $(\|\cdot\|_n : n \in \mathbb{N})$ *is a multi-norm on the family* $\{E^n : n \in \mathbb{N}\}$ *;*

(b) $\|a \cdot x\|_m \le \|a\| \, \|x\|_n$ *for each row-special matrix* $a \in \mathbb{M}_{m,n}$*, each* $x \in E^n$*, and each* $m, n \in \mathbb{N}$ *;*

(c) $\|a \cdot x\|_m \le \|a\| \, \|x\|_n$ *for each matrix* $a \in \mathbb{M}_{m,n}$*, each* $x \in E^n$*, and each* $m, n \in \mathbb{N}$*.* □

There is a similar reformulation of the definition of a dual multi-norm.

2. Lecture - The sequence $(\varphi_n(E))$

2.1. An associated sequence. The sequence defined below measures the 'rate of growth' of a multi-norm.

Definition 2.1. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space. For* $n \in \mathbb{N}$*, set*

$$\varphi_n(E) = \sup \{\|(x_1, \ldots, x_n)\|_n : x_1, \ldots, x_n \in E_{[1]}\} .$$

Note that $(\varphi_n(E))$ is not intrinsic to E; it depends on the multi-norm, and so, strictly, we should write $\varphi_n((E^n, \|\cdot\|_n))$ for $\varphi_n(E)$. The sequence $(\varphi_n(E))$ is increasing and convex.

2.2. The minimum multi-norm.

Definition 2.2. *Let* $(E, \|\cdot\|)$ *be a normed space. For* $n \in \mathbb{N}$*, define* $\|\cdot\|_n$ *on* E^n *by*

$$\|(x_1, \ldots, x_n)\|_n^{\min} = \max_{i=1,\ldots,n} \|x_i\| \quad (x_1, \ldots, x_n \in E) .$$

This gives the minimum multi-norm.

Two multi-norms $(\|\cdot\|_n^1 : n \in \mathbb{N})$ and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ are *equivalent* if there exist constants C_1 and C_2 such that

$$C_1 \|(x_1, \ldots, x_n)\|_n^2 \le \|(x_1, \ldots, x_n)\|_n^1 \le C_2 \|(x_1, \ldots, x_n)\|_n^2$$

for all $x_1, \ldots, x_n \in E$ and $n \in \mathbb{N}$.

Proposition 2.3. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space such that* E *is finite-dimensional. Then* $(\|\cdot\|_n : n \in \mathbb{N})$ *is equivalent to the minimum multi-norm.* □

In fact, there are always multi-norms that are not equivalent to the minimum multi-norm when is E is infinite-dimensional.

2.3. The maximum multi-norm. Let $(E, \|\cdot\|)$ be a normed space. It follows from Lemma 1.5 that there is also a maximum multi-norm $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$.

We have

$$\|(x_1, \ldots, x_n)\|_n^{\max} \leq \sum_{j=1}^{n} \|x_j\| \ .$$

But the right-hand side does **NOT** define a multi-norm. (In fact it is a dual multi-norm.)

The sequence associated with the maximum multi-norm is denoted by $(\varphi_n^{\max}(E))$. This sequence **is** intrinsic to E. We shall calculate it for some examples soon.

PROPOSITION 2.4. *Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be two linearly homeomorphic Banach spaces. Then*

$$\varphi_n^{\max}(F) \leq d(E, F)\varphi_n^{\max}(E) \quad (n \in \mathbb{N})\,. \qquad \square$$

2.4. Summing norms. We recall some results on summing norms; for nice introductions to this theory, see [**4**] and [**5**].

Let E be a normed space, let $n \in \mathbb{N}$, let $x_1, \ldots, x_n \in E$, and take $p \geq 1$. We define the **weak p-summing norm**:

$$\mu_{p,n}(x_1, \ldots, x_n) = \sup \left\{ \left(\sum_{j=1}^{n} |\langle x_j, \lambda \rangle|^p \right)^{1/p} : \lambda \in E'_{[1]} \right\} .$$

We note that

$$\mu_{1,n}(x_1, \ldots, x_n) = \sup \left\{ \left\| \sum_{j=1}^{n} \zeta_j x_j \right\| : \zeta_1, \ldots, \zeta_n \in \mathbb{T} \right\} .$$

The next theorem shows how these norms fit into our scenario.

THEOREM 2.5. *Let $(E, \|\cdot\|)$ be a normed space. Then $(\mu_{1,n} : n \in \mathbb{N})$ is a dual multi-norm on $\{E^n : n \in \mathbb{N}\}$, and*

$$\mu_{1,n}(x_1, \ldots, x_n) \leq \|(x_1, \ldots, x_n)\|_n \quad (x_1, \ldots, x_n \in E)$$

whenever $(\|\cdot\|_n : n \in \mathbb{N})$ is a dual multi-norm on $\{E^n : n \in \mathbb{N}\}$. $\square$

Thus $(\mu_{1,n} : n \in \mathbb{N})$ is the minimum dual multi-norm on the family $\{E^n : n \in \mathbb{N}\}$.

2.5. Summing constants. The following definition is standard; see [**5**].

Definition 2.6. *Let E and F be normed spaces, let $n \in \mathbb{N}$, and take $p \geq 1$. Then the* summing constants *of an operator $T \in \mathcal{B}(E, F)$ are the numbers*

$$\pi_p^{(n)}(T) := \sup \left\{ \left(\sum_{j=1}^{n} \|Tx_j\|^p \right)^{1/p} : \mu_{p,n}(x_1, \dots, x_n) \leq 1 \right\}.$$

Further, $\pi_p^{(n)}(E) = \pi_p^{(n)}(I_E)$; these are the summing constants *of the normed space E.*

In particular,

$$\pi_1^{(n)}(E) = \sup \left\{ \sum_{j=1}^{n} \|x_j\| : \left\| \sum_{j=1}^{n} \zeta_j x_j \right\| \leq 1 \ \ (\zeta_1, \dots, \zeta_n \in \mathbb{T}) \right\}.$$

Spaces with the following property are important in Banach space theory.

Definition 2.7. *A Banach space E has the* Orlicz property *if*

$$\sup_{n \in \mathbb{N}} \left\{ \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{1/2} : x_1, \dots, x_n \in E,\ \mu_{1,n}(x_1, \dots, x_n) \leq 1 \right\} < \infty.$$

Theorem 2.8. *Let (Ω, μ) be a σ-finite measure space, where μ is a positive measure, and take $q \in [1, 2]$. Then the Banach space $L^q(\Omega, \mu)$ has the Orlicz property.* □

Let $q \in [1, 2]$. The Orlicz constant associated with the space $\ell^{\,q}$ is denoted by C_q, where we know that $C_2 = 1$ and that $C_1 \leq \sqrt{2}$ [**4, 5**]. Thus

$$\pi_1^{(n)}(\ell^{\,q}) \leq C_q \sqrt{n} \quad (n \in \mathbb{N}).$$

In particular,

$$\pi_1^{(n)}(\ell^{\,2}) \leq \sqrt{n} \quad (n \in \mathbb{N}).$$

It would be interesting to find the exact values of $\pi_1^{(n)}(\ell_m^{\,p})$ for each $m, n \in \mathbb{N}$ and $p \in [1, \infty]$.

THEOREM 2.9. *Let* $(E, \|\cdot\|)$ *be a normed space. Then*

$$\begin{aligned}&\|(x_1, \dots, x_n)\|_n^{\max}\\ &\quad = \sup\left\{\left|\sum_{j=1}^n \langle x_j,\, \lambda_j\rangle\right| : \lambda_1, \dots, \lambda_n \in E',\, \mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1\right\}\\ &\quad = \sup\left\{\sum_{j=1}^n |\langle x_j,\, \lambda_j\rangle| : \lambda_1, \dots, \lambda_n \in E',\, \mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1\right\}\end{aligned}$$

for each $n \in \mathbb{N}$ *and* $x_1, \dots, x_n \in E$. *Further, the dual of* $\|\cdot\|_n^{\max}$ *is* $\mu_{1,n}$ *for each* $n \in \mathbb{N}$, *and* $\varphi_n^{\max}(E)$ *is equal to*

$$\sup\left\{\sum_{j=1}^n \|\lambda_j\| : \lambda_1, \dots, \lambda_n \in E',\, \mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1\right\}.$$

Thus

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(E').$$

□

THEOREM 2.10. *Let* $(E, \|\cdot\|)$ *be a normed space. Then*

$$(\mu'_{1,n} : n \in \mathbb{N})$$

is the maximum multi-norm on the family $\{(E')^n : n \in \mathbb{N}\}$. □

In **summary**, we have the following.

Let $(E, \|\cdot\|)$ be a normed space. Then the maximum multi-norm on the family $\{E^n : n \in \mathbb{N}\}$ is denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$. The dual of this multi-norm is the minimum dual multi-norm on the family $\{(E')^n : n \in \mathbb{N}\}$, and this is exactly the multi-norm $(\mu_{1,n} : n \in \mathbb{N})$ on $\{(E')^n : n \in \mathbb{N}\}$, and this is the weak 1-summing norm. The dual of the minimum dual multi-norm on the family $\{E^n : n \in \mathbb{N}\}$ is the maximum multi-norm on $\{(E')^n : n \in \mathbb{N}\}$.

Combining these remarks, we have the following consequence.

COROLLARY 2.11. *Let* $(E, \|\cdot\|)$ *be a normed space. Then the second dual of the maximum multi-norm* $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ *on* $\{E^n : n \in \mathbb{N}\}$ *is the maximum multi-norm on* $\{(E'')^n : n \in \mathbb{N}\}$. □

THEOREM 2.12. *Let* $(E, \|\cdot\|)$ *be a normed space. Then*

$$(E^n, \|\cdot\|_n^{\max}) \cong (\ell_n^\infty \otimes E, \pi)$$

for each $n \in \mathbb{N}$. □

Let S_F denote the unit sphere of a normed space F. (We shall suppose henceforth that $F \neq \{0\}$, so that $S_F \neq \emptyset$.)

Definition 2.13. *Let $n \in \mathbb{N}$, and let $(F, \|\cdot\|)$ be a normed space. Then $c_n(F)$ is*

$$\inf\{\sup\{\|\zeta_1\lambda_1 + \cdots + \zeta_n\lambda_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} : \lambda_1, \dots, \lambda_n \in S_F\}\,.$$

Theorem 2.14. *Let $(F, \|\cdot\|)$ be a normed space, and let $n \in \mathbb{N}$. Then*

$$\pi_1^{(n)}(F) \,\cdot\, c_n(F) \geq n\,,$$

and so $\varphi_n^{\max}(E) \geq n/c_n(E')$ for each normed space E.

Proof. Let $\overline{\pi}_1^{(n)}(F)$ be the version of $\pi_1^{(n)}(F)$ in which we require, further, that $\|x_1\| = \cdots = \|x_n\|$ in the definition. Then it is clear that $\overline{\pi}_1^{(n)}(F) \leq \pi_1^{(n)}(F)$ and that $\overline{\pi}_1^{(n)}(F) \,\cdot\, c_n(F) = n$. □

We **conjecture** that, for each normed space E, or perhaps for a reasonable class of 'well-behaved' spaces E, there is a constant C_E independent of n such that $\varphi_n^{\max}(E) \leq C_E n/c_n(E')$ $(n \in \mathbb{N})$. Is this true for all spaces with the Orlicz property?

2.6. The function $\varphi_n^{\max}$ for some examples.

We can calculate $\varphi_n^{\max}(E)$ for most standard Banach spaces E; here are some examples; more are given in [**2**].

Theorem 2.15. (i) *For each $p \in [1, 2]$, we have*

$$\varphi_n^{\max}(\ell_n^p) = \varphi_n^{\max}(\ell^p) = n^{1/p} \quad (n \in \mathbb{N})\,;$$

(ii) *For each $p \in [2, \infty]$, we have*

$$\sqrt{n} \leq \varphi_n^{\max}(\ell_n^p) \leq \varphi_n^{\max}(\ell^p) \leq C_q\sqrt{n} \quad (n \in \mathbb{N})\,,$$

where q is the conjugate index to p. □

We do not know the exact best constant that could replace C_q in the above inequality.

2.7. A lower bound for $\varphi_n^{\max}(E)$.

We shall use a famous theorem of Dvoretzky, sometimes called *the theorem on almost spherical sections.*

Theorem 2.16. *For each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m = m(n, \varepsilon)$ in $\mathbb{N}$ such that, for each normed space E with $\dim E \geq m$, there is an n-dimensional subspace L of E such that $d(L, \ell_n^2) < 1 + \varepsilon$.* □

Theorem 2.17. *Let E be an infinite-dimensional normed space. Then $c_n(E) \leq \sqrt{n}$ and $\varphi_n^{\max}(E) \geq \sqrt{n}$ for each $n \in \mathbb{N}$.*

PROOF. Fix $n \in \mathbb{N}$. By Dvoretzky's theorem, for each $\varepsilon > 0$, there is an n-dimensional subspace L in E such that $d(L, \ell_n^2) < 1 + \varepsilon$. Certainly $c_n(E) \leq c_n(L)$.

Since $d(L, \ell_n^2) < 1 + \varepsilon$, it follows from the definition of $c_n(L)$ that $c_n(L) \leq (1+\varepsilon)^2 c_n(\ell_n^2)$. But $c_n(\ell_n^2) = \sqrt{n}$, and so $c_n(E) \leq (1+\varepsilon)^2 \sqrt{n}$. This holds for each $\varepsilon > 0$, and hence $c_n(E) \leq \sqrt{n}$.

That $\varphi_n^{\max}(E) \geq \sqrt{n}$ for each $n \in \mathbb{N}$ follows immediately from Theorem 2.14. □

2.8. Another characterization of multi-norms. Let E be a normed space. Form the algebraic tensor product $c_0 \otimes E$. A *cross-norm* on $c_0 \otimes E$ is a norm $\|\cdot\|$ such that $\|a \otimes x\| = \|a\| \, \|x\|$ for each $a \in c_0$ and $x \in E$. This norm is a c_0*-norm* if, further, $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ by $\|T\|$ for each compact operator T on c_0.

THEOREM 2.18. (Daws) *Let E be a normed space. Then there is a canonical bijection between the family of multi-norms based on E and the family of c_0-norms on $c_0 \otimes E$.*

PROOF. Start from a c_0-norm $\|\cdot\|$ on $c_0 \otimes E$. Define

$$\|(x_1, \dots, x_n)\|_n = \left\| \sum_{j=1}^{n} \delta_j \otimes x_j \right\| \quad (x_1, \dots, x_n \in E)\,.$$

□

In the above correspondence, the minimum and maximum multi-norms correspond to the injective and projective tensor products on $c_0 \otimes E$, respectively.

3. Lecture - Examples of multi-norms

3.1. The standard (p, q)-multi-norm.

EXAMPLE 3.1. Let $\Omega = (\Omega, \mu)$ be a σ-finite measure space, where μ is a positive measure, and take p, q with $1 \leq p \leq q < \infty$. We consider the Banach space $E = L^p(\Omega)$, with the norm

$$\|f\| = \left(\int_\Omega |f|^p \right)^{1/p} = \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{1/p} \quad (f \in E)\,.$$

For a measurable subset X of Ω, we write r_X for the seminorm on E specified by

$$r_X(f) = \left(\int_X |f|^p \right)^{1/p} \quad (f \in E)\,.$$

Take $n \in \mathbb{N}$. For each partition $\mathbf{X} = \{X_1, \dots, X_n\}$ of Ω into measurable subsets and each $f_1, \dots, f_n \in E$, we set

$$\begin{aligned} r_{\mathbf{X}}((f_1, \dots, f_n)) &= \left(r_{X_1}(f_1)^q + \cdots + r_{X_n}(f_n)^q\right)^{1/q} \\ &= \left(\left(\int_{X_1} |f_1|^p\right)^{q/p} + \cdots + \left(\int_{X_n} |f_n|^p\right)^{q/p}\right)^{1/q}, \end{aligned}$$

so that $r_{\mathbf{X}}$ is a seminorm on E^n.

Finally, define

$$\|(f_1, \dots, f_n)\|_n^{(p,q)} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \dots, f_n)) \quad (f_1, \dots, f_n \in E),$$

where the supremum is taken over all such families $\mathbf{X}$. Then $\|\cdot\|_n$ is a norm on E^n.

It is easily checked that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on the family $\{E^n : n \in \mathbb{N}\}$. It is the *standard* (p, q)*-multi-norm.* $\square$

PROPOSITION 3.2. *In each of the above cases, the standard* $(1, 1)$*-multi-norm on* $\{L^1(\Omega)^n : n \in \mathbb{N}\}$ *is equal to the maximum multi-norm. However this is not true for the standard* (p, p)*-multi-norm on* ℓ^p *for any* $p > 1$. $\square$

EXAMPLE 3.3. Let Ω be a non-empty, locally compact space. We now denote by $M(\Omega)$ the Banach space of all complex-valued, regular Borel measures on Ω. Take $q \geq 1$.

For each partition $\mathbf{X} = \{X_1, \dots, X_n\}$ of Ω into measurable subsets and each $\mu_1, \dots, \mu_n \in M(\Omega)$, we set

$$r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) = \left(\|\mu_1 \mid X_1\|^q + \cdots + \|\mu_n \mid X_n\|^q\right)^{1/q},$$

so that $r_{\mathbf{X}}$ is a seminorm on $M(\Omega)^n$ and

$$r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) \leq \left(\|\mu_1\|^q + \cdots + \|\mu_n\|^q\right)^{1/q} \quad (\mu_1, \dots, \mu_n \in M(\Omega)).$$

Finally, we define

$$\|(\mu_1, \dots, \mu_n)\|_n = \sup_{\mathbf{X}} r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) \quad (\mu_1, \dots, \mu_n \in M(\Omega)),$$

where the supremum is taken over all such families $\mathbf{X}$. Then $\|\cdot\|_n$ is a norm on $M(\Omega)^n$, and it is again easily checked that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{M(\Omega)^n : n \in \mathbb{N}\}$. It is the *standard* $(1, q)$*-multi-norm.* $\square$

Let Ω be a non-empty, locally compact space, and set $E = L^1(\Omega)$. Then $E' = L^\infty(\Omega)$, a commutative C^*-algebra, and so this is $C(\widetilde{\Omega})$ for some compact space $\widetilde{\Omega}$. Thus $E'' = M(\widetilde{\Omega})$. Start with the standard

$(1,q)$-multi-norm on $\{L^1(\Omega)^n : n \in \mathbb{N}\}$: then we can compare the standard $(1,q)$-multi-norm on $M(\widetilde{\Omega})$ with the second dual of the standard $(1,q)$-multi-norm on $L^1(\Omega)$. In fact, happily they are the same - but this seems to be quite hard; the proof of this also uses the principle of local reflexivity.

3.2. The Hilbert multi-norm.

EXAMPLE 3.4. Let H be a Hilbert space. Then H can be represented as the Banach space $\ell^2(S)$ for a set S of vectors in H. Thus from each such set S and each q with $2 \leq q < \infty$, we obtain the standard $(2,q)$-multi-norm $(\|\cdot\|_n^{(2,p)} : n \in \mathbb{N})$ on $\{H^n : n \in \mathbb{N}\}$, as above.

Now we introduce another multi-norm on this family; it is the maximum that we can obtain by considering all such representations of H. Take $n \in \mathbb{N}$. For each family $\mathbf{H} = \{H_1, \dots, H_n\}$ such that $H = H_1 \perp \cdots \perp H_n$ (which means that $H = H_1 \oplus \cdots \oplus H_n$ and that the closed subspaces $H_1, \dots, H_n$ of H are pairwise orthogonal), set

$$\left\{\begin{array}{lcl} r_{\mathbf{H}}((x_1, \dots, x_n)) & = & \left(\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2\right)^{1/2} \\ & = & \|P_1 x_1 + \cdots + P_n x_n\| \end{array}\right.$$

for $x_1, \dots, x_n \in H$, where $P_i : H \to H_i$ for $i = 1, \dots, n$ is the orthogonal projection, and then set

$$\|(x_1, \dots, x_n)\|_n^H = \sup_{\mathbf{H}} r_{\mathbf{H}}((x_1, \dots, x_n)) \quad (x_1, \dots, x_n \in H)\,,$$

where the supremum is taken over all such families $\mathbf{H}$.

It is easily checked that $(\|\cdot\|_n^H : n \in \mathbb{N})$ is a multi-norm on the family $\{H^n : n \in \mathbb{N}\}$. This is the *Hilbert multi-norm* on the family $\{H^n : n \in \mathbb{N}\}$. □

PROPOSITION 3.5. *Let H be a Hilbert space, and let $n \in \mathbb{N}$. Then*

$$\|(x_1, \dots, x_n)\|_n^H = \sup\{|\alpha_1[e_1, x_1] + \cdots + \alpha_n[e_n, x_n]|\}$$

for $x_1, \dots, x_n \in H$, where the supremum is taken over all orthonormal sets $\{e_1, \dots, e_n\}$ in H and all $(\alpha_1, \dots, \alpha_n) \in (\ell_n^2)_{[1]}$. □

Question Is the Hilbert multi-norm the maximum multi-norm on the family $\{H^n : n \in \mathbb{N}\}$? This seemed to be very likely because I could not think of a bigger one. However it seems to be rather a hard question.

In fact it can be reduced to a question about Hilbert spaces that does not mention multi-norms; possibly the answer to this question is already known.

Let H be a Hilbert space. Then the closed unit ball of the dual of $(H^n, \|\cdot\|_n^H)$ is described as follows. Set

$$S := \bigcup \left\{ (\alpha_1 e_1, \dots, \alpha_n e_n) : \sum_{j=1}^{n} |\alpha_j|^2 \leq 1 \right\} ,$$

where the union is taken over all orthonormal subsets $\{e_1, \dots, e_n\}$ of H. The required unit ball is the weak-$*$-closed convex hull of S, call it K.

On the other hand, the closed unit ball of the dual of $(H^n, \|\cdot\|_n^{\max})$ is

$$\{y = (y_1, \dots, y_n) \in H^n : \mu_{1,n}(y_1, \dots, y_n) \leq 1\} ;$$

this set, temporarily called L, is equal to

$$\{y = (y_1, \dots, y_n) \in H^n : \|\zeta_1 y_1 + \cdots + \zeta_n y_n\| \leq 1 \ \ (\zeta_1, \dots, \zeta_n \in \mathbb{T})\} .$$

Since $\|\cdot\|_n^H \leq \|\cdot\|_n^{\max}$, necessarily $K \subset L$.

To establish the equality of the two multi-norms, we need to show that $L \subset K$ for each (implicit) $n \in \mathbb{N}$. In fact we need

$$\operatorname{ex} L \subset \operatorname{ex} K = S \quad (n \in \mathbb{N}) ,$$

where 'ex' denotes the set of extreme points of a convex set. Is this always the case? This question does not mention multi-norms. For each $n \in \mathbb{N}$, it is sufficient to consider Hilbert spaces of dimension n. Towards this, I know the following.

THEOREM 3.6. *Let H be a Hilbert space of dimension n.*

(i) *Suppose that $n = 2$. Then* $\operatorname{ex} L \subset S$.

(ii) *Suppose that $n = 3$ and H is a* **real** *Hilbert space. Then this fails.*

(iii) (Pham) *Suppose that $n = 3$ and H is complex. Then* $\operatorname{ex} L \subset S$.

(iv) (Daws) *There is a universal constant C with $C\|\cdot\|_n^H \geq \|\cdot\|_n^{\max}$, and so the Hilbert multi-norm is equivalent to the maximum multi-norm.* □

(In fact the best C in (iv) that we know involves Grothendieck's constant.)

3.3. The lattice multi-norm.

EXAMPLE 3.7. Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$, set

$$\|(x_1, \dots, x_n)\|_n = \|\,|x_1| \vee \cdots \vee |x_n|\,\| \quad (x_1, \dots, x_n \in E) .$$

It is easy to check that $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-Banach space. The sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is called the *lattice multi-norm*. On the other hand, by setting

$$\|(x_1, \ldots, x_n)\|_n = \|\, |x_1| + \cdots + |x_n|\, \| \quad (x_1, \ldots, x_n \in E)\,,$$

we obtain a dual multi-norm, the *dual lattice multi-norm*.

It is fairly straightforward to check that the dual of the latttice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the dual lattice multi-norm on the family $\{(E')^n : n \in \mathbb{N}\}$, and that the dual of the dual latttice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the lattice multi-norm on $\{(E')^n : n \in \mathbb{N}\}$.

More specifically, we have the following examples:

1) Let $E = C(\Omega)$ for a compact space Ω. Then the lattice multi-norm is just the minimum multi-norm.

2) Let $E = M(\Omega)$ for a measure space Ω. Then the lattice multi-norm is just the standard $(1,1)$-multi-norm.

3) Let $E = L^p(\Omega)$ for a measure space Ω and $p \geq 1$. Then the lattice norm is the standard (p,p)-multi-norm. □

4. Lecture - Multi-bounded linear operators

4.1. Topological linear spaces and multi-norms. There is a theory - it generalizes that of multi-normed spaces to give a theory of *multi-topological linear spaces*. It is specified in [**2**], but it is more-or-less what one would expect, and so we do not give it here. It includes the concept of *multi-null sequence*, for which we write

$$\operatorname*{Lim}_{i\to\infty} x_i = 0 \quad \text{in } E\,.$$

Having obtained a general concept, we can reduce back to a multi-normed space. We obtain the following.

Theorem 4.1. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Take $(x_i) \in E^{\mathbb{N}}$. Then (x_i) is a multi-null sequence in E if and only if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{k\in\mathbb{N}} \|(x_{n+1}, \ldots, x_{n+k})\|_k < \varepsilon \quad (n \geq n_0)\,.$$

□

There is a version of Kolmogorov's theorem.

Example 4.2. Let (α_i) be a fixed element of $\mathbb{C}^{\mathbb{N}}$, and set

$$x_i = \alpha_i \delta_i \quad (i \in \mathbb{N})\,.$$

(i) Let E be one of the Banach spaces ℓ^p (for $p \geq 1$) or c_0. Let $(\|\cdot\|_n : n \in \mathbb{N})$ be the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$. Then

(x_i) is a multi-null sequence in E if and only if $\lim_{i\to\infty}\alpha_i = 0$, i.e., if and only if $(\alpha_i) \in c_0$. This is independent of the choice of the space E.

(ii) Let $E = \ell^p$ (where $p \geq 1$), and let $(\|\cdot\|_n : n \in \mathbb{N})$ be the standard (p,p)-multi-norm on $\{E^n : n \in \mathbb{N}\}$. Then (x_i) is a multi-null sequence in E if and only if

$$\lim_{n\to\infty}\left(\sum_{i=n}^{\infty}|\alpha_i|^p\right)^{1/p} = 0\,,$$

i.e., if and only if $(\alpha_i) \in \ell^p$. $\square$

4.2. Multi-null sequences and order-convergence. A new theory should reduce to something familiar when we restrict to a familiar situation; we shall consider this first in the context of lattice multi-norms.

Let E be a Banach lattice, as above, and let (x_n) be a sequence in E. Recall that (x_n) is *order-null* if there is a sequence (u_n) in $E_{\mathbb{R}}$ such that $u_n \downarrow 0$ and $|x_n| \leq u_n \ (n \in \mathbb{N})$. The lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ was defined by

$$\|(x_1,\dots,x_n)\|_n = \|\,|x_1| \vee \cdots \vee |x_n|\,\| \quad (x_1,\dots,x_n \in E)$$

for each $n \in \mathbb{N}$. We shall consider multi-null sequences with respect to this multi-norm.

THEOREM 4.3. *Let E be a Banach lattice. Then each multi-null sequence in E is order-null in E.* $\square$

THEOREM 4.4. *Let $(E, \|\cdot\|)$ be a Banach lattice. Then each order-null sequence in E is multi-null in E if and only if the norm is σ-order-continuous, i.e., $\|x_n\| \downarrow 0$ whenever (x_n) is a sequence in E such that $x_n \downarrow 0$.* $\square$

For example, this applies to the spaces $L^p(\Omega)$ with the standard (p,p)-multi-norm when $p \geq 1$.

4.3. Multi-continuous operators. Here is the obvious definition.

DEFINITION 4.5. *A linear map between two multi-topological linear spaces is* multi-continuous *if it maps multi-null sequences into multi-null sequences.*

4.4. Multi-bounded sets and maps. There is a general concept of a *multi-bounded set* in a multi-topological linear space. A linear map between such spaces is *multi-bounded* if it takes multi-bounded sets into multi-bounded sets. Again we give the definition just in the context of multi-normed spaces.

DEFINITION 4.6. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let B be a subset of E. Then B is* multi-bounded *in E if*

$$c_B := \sup\{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in B,\, n \in \mathbb{N}\} < \infty\,.$$

For example, in most Banach lattices, a set is multi-bounded in the lattice multi-norm if and only if it is order-bounded in the lattice.

The space of multi-bounded maps from E to F is denoted by $\mathcal{M}(E, F)$. Clearly it is a linear subspace of $\mathcal{B}(E, F)$.

DEFINITION 4.7. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces, and let T be a multi-bounded linear operator from E to F. Then*

$$\|T\|_{mb} = \sup\{c_{T(B)} : c_B \le 1\}\,.$$

The map T is a multi-contraction *if $\|T\|_{mb} \le 1$, and T is a* multi-isometry *if T is an isometry onto a closed subspace $T(E)$ of F and if $T \in \mathcal{M}(E, T(E))$ and $T^{-1} \in \mathcal{M}(T(E), E)$ are both multi-contractions.*

It is clear that $\|\cdot\|_{mb}$ is a norm on the space $\mathcal{M}(E, F)$.

Indeed, for $n \in \mathbb{N}$, set

$$p_n(T) = \sup\{\|(Tx_1, \dots, Tx_n)\|_n : \|(x_1, \dots, x_n)\|_n \le 1\}\,.$$

Then $(p_n(T) : n \in \mathbb{N})$ is an increasing sequence with

$$\|T\|_{mb} = \lim_{n\to\infty} p_n(T)\,.$$

Clearly we have

$$\|T\|_{mb} = \sup_n\ \sup\left\{\frac{\|(Tx_1, \dots, Tx_n)\|_n}{\|(x_1, \dots, x_n)\|_n} : (x_1, \dots, x_n) \ne 0\right\} < \infty\,.$$

The next basic proposition shows that we are establishing a multi-version of another very basic result in functional analysis.

THEOREM 4.8. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces. Then a map $T \in \mathcal{L}(E, F)$ is multi-continuous if and only if it is multi-bounded.* □

In the following theorem, $\mathcal{N}(E, F)$ denotes the space of all nuclear operators from E to F; the nuclear norm is denoted by ν.

THEOREM 4.9. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *and* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be multi-normed spaces, with* F *a Banach space. Then*

$$(\mathcal{M}(E,F), \|\cdot\|_{mb})$$

is a Banach space. Further,

$$(\mathcal{N}(E,F), \nu) \subset (\mathcal{M}(E,F), \|\cdot\|_{mb}) \subset (\mathcal{B}(E,F), \|\cdot\|)\,,$$

and the natural embeddings are contractions. □

A key point in standard functional analysis is that, if E and F are Banach spaces, then so is $\mathcal{B}(E,F)$; we stay in the category when we take morphisms. We would like to do the same in the multi-context. Happily, this works.

DEFINITION 4.10. *Let*

$$((E^n, \|\cdot\|_n) : n \in \mathbb{N}) \quad \text{and} \quad ((F^n, \|\cdot\|_n) : n \in \mathbb{N})$$

be multi-normed spaces, and let $n \in \mathbb{N}$ *and* $T_1, \dots, T_n \in \mathcal{M}(E,F)$. *Then*

$$\|(T_1, \dots, T_n)\|_{mb,n} = \sup\{c_{T_1(B)\cup\cdots\cup T_n(B)} : c_B \leq 1\}\,.$$

The supremum is always finite.

More explicitly, choose $k_1, \dots, k_n \in \mathbb{N}$, and set $k = k_1 + \cdots + k_n$. Then take $x_1, \dots, x_k \in E$ with $\|(x_1, \dots, x_k)\|_k \leq 1$, and consider the element $Tx \in F^k$ specified by

$$\begin{aligned} Tx = (&T_1x_1, \dots, T_1x_{k_1}, \\ &T_2x_{k_1+1}, \dots, T_2x_{k_1+k_2}, \dots, T_nx_{k_1+k_2+\cdots+k_{n-1}+1}, \dots, T_nx_k)\,. \end{aligned}$$

We see that

$$\|(T_1, \dots, T_n)\|_{mb,n} = \sup\{\|Tx\|_k\}\,,$$

where the supremum is taken over all choices satisfying the prescribed conditions. In particular, $\|(T_1, \dots, T_n)\|_{mb,n} \geq \max\|T_i\|$.

THEOREM 4.11. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *and* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be multi-normed spaces. Then each* $\|\cdot\|_{mb,n}$ *is a norm on the linear space* $\mathcal{M}(E,F)^n$, *and*

$$((\mathcal{M}(E,F)^n, \|\cdot\|_{mb,n}) : n \in \mathbb{N})$$

is a multi-normed space with $\|T\|_{mb,1} = \|T\|_{mb}$; *it is a multi-Banach space in the case where* F *is a Banach space.* □

4.5. Examples.

EXAMPLE 4.12. Let H be the Hilbert space $\ell^2(\mathbb{N})$, with the standard $(2,2)$-multi-norm.

Consider the system of vectors $(x_r^s : r = 1, \dots s, s \in \mathbb{N})$ in H defined as follows: $x_r^s(k) = 0$ except when $k \in \{2^{s-1}, \dots, 2^s - 1\}$; at the 2^{s-1} numbers k in the set $\{2^{s-1}, \dots, 2^s - 1\}$, $x_r^s(k) = \pm 1/\sqrt{2^{s-1}}$, the values ± 1 being chosen so that $[x_{r_1}^s, x_{r_2}^s] = 0$ when $r_1, r_2 = 1, \dots, s$ and $r_1 \neq r_2$. Such a choice is clearly possible. Then

$$S := \{x_r^s : r = 1, \dots, s, s \in \mathbb{N}\}$$

is an orthonormal set in H. Order the set S as (y_n) by using the lexicographic order on the pairs (s, r).

Let $(\alpha_i) \in \ell^\infty$. We define an operator $T \in \mathcal{B}(H)$ by setting

$$Tx_r^s = \alpha_s \delta_n \quad \text{when} \quad x_r^s = y_n\,.$$

It is clear that, in the case where $(\alpha_i) \in c_0$, we have $T \in \mathcal{K}(H)$.

For $k \in \mathbb{N}$, set $N_k = \sum_{i=1}^k i = k(k+1)/2$. We see that

$$\|(y_1, y_2, \dots, y_{N_k})\|_{N_k}^2 = k\,.$$

However

$$\begin{aligned} \|(Ty_1, Ty_2, \dots, Ty_{N_k})\|_{N_k}^2 &= \|(\alpha_1\delta_1, \alpha_2\delta_2, \alpha_2\delta_3, \alpha_3\delta_4, \dots, \alpha_k\delta_{N_k})\|_{N_k}^2 \\ &= \sum_{i=1}^k i\,|\alpha_i|^2\,. \end{aligned}$$

Now take $\gamma \in (0, 1/2)$, and set $\alpha_i = i^{-\gamma}$ $(i \in \mathbb{N})$, so that $(\alpha_i) \in c_0$. Then

$$\sum_{i=1}^k i\,|\alpha_i|^2 = \sum_{i=1}^k i^{1-2\gamma} \geq \int_1^k t^{1-2\gamma}\,\mathrm{d}t \geq \frac{1}{2-2\gamma}(k^{2-2\gamma} - 1)\,.$$

Thus

$$\frac{\|(Ty_1, Ty_2, \dots, Ty_{N_k})\|_{N_k}}{\|(y_1, y_2, \dots, y_{N_k})\|_{N_k}} \geq ck^{(1-2\gamma)/2}$$

for a constant $c > 0$. Since $\gamma < 1/2$, we have $T \notin \mathcal{M}(H)$.

We have shown that $\mathcal{K}(H) \not\subset \mathcal{M}(H)$. In particular, $\mathcal{M}(H) \subsetneq \mathcal{B}(H)$. Since $I_H \in \mathcal{M}(H)$, we have $\mathcal{M}(H) \not\subset \mathcal{K}(H)$. (Recall again that the space $\mathcal{M}(H)$ given here depends on the choice of the multi-norm.)

What is the characterization of $\mathcal{M}(H)$ in this case? □

EXAMPLE 4.13. Now let H be the Hilbert space $\ell^2(\mathbb{N})$, and give the family $\{H^n : n \in \mathbb{N}\}$ the Hilbert multi-norm. By the theorem of Daws given above, the Hilbert multi-norm is equivalent to the maximum multi-norm, and so it is easy to see that $\mathcal{M}(H) = \mathcal{B}(H)$. Thus we can give the family $\{\mathcal{B}(H)^n : n \in \mathbb{N}\}$ the structure of a multi-normed space. However, the multi-normed structure is that of the minimum multi-norm, so this is not very interesting. $\square$

Example 4.14. One might guess that a form of Banach's isomorphism theorem would hold for multi-bounded operators. However this is not the case.

Let $E = \ell^1$. Then $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space for the standard $(1,1)$-multi-norm. In this case,

$$\|(\delta_1, \dots, \delta_n)\|_n = n \quad (n \in \mathbb{N}) .$$

However, let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be the multi-normed space formed from E by taking the minimum multi-norm $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$. Then

$$\|(\delta_1, \dots, \delta_n)\|_n^{\min} = 1 \quad (n \in \mathbb{N}) .$$

This shows that the identity map I_E on E, regarded as map from E to F belongs to $\mathcal{M}(E, F)$, but that $I_E : F \to E$ is not multi-bounded. Indeed $\mathcal{M}(E, F) = \mathcal{B}(E, F)$.

We shall now identify $\mathcal{M}(F, E)$. Take $T \in \mathcal{M}(F, E)$. The unit ball $F_{[1]}$ of F is multi-bounded, and so $T(F_{[1]})$ is multi-bounded in E. Since F is monotonically bounded (see below), it follows that $F_{[1]}$ is order-bounded in $E = \ell^1$, and so there exists $x = (x_n) \in \ell^1$ with

$$|T(y)_i| \le x_i \quad (i \in \mathbb{N})$$

for each $y \in F_{[1]}$; further, $\sum_{i=1}^{\infty} x_i \ge \|T\|_{mb}$. For $i \in \mathbb{N}$, let $\pi_i : z \mapsto z_i \delta_i$ be the rank-one operator on ℓ^1, and set $T_i = \pi_i \circ T = \delta_i \otimes T'(\delta_i)$, so that

$$\nu(T_i) = \|T'(\delta_i)\| \, \|\delta_i\| \le \|T\| ,$$

where ν again denotes the nuclear norm. Then $T = \sum_{i=1}^{\infty} x_i T_i$, and hence $\nu(T) = \sum_{i=1}^{\infty} x_i \|T\| < \infty$. Thus $T \in \mathcal{N}(F, E)$.

In summary, we have

$$\mathcal{M}(E, F) = \mathcal{B}(E, F) \quad \text{and} \quad \mathcal{M}(F, E) = \mathcal{N}(F, E)$$

in this case $\square$

However we do not know what happens when the two multi-normed components are the same; maybe there is a multi-Banach isomorphism theorem in this situation?

4.6. Multi-bounded operators on Banach lattices. Again we would like to identify the multi-bounded operators in a familiar situation.

A Banach lattice $(E, \|\cdot\|)$ is *monotonically bounded* if every increasing net in the unit ball of $E_{\mathbb{R}}$ is bounded above, and it is *Dedekind complete* if every set in $E_{\mathbb{R}}$ which is bounded above has a supremum.

Let E and F be real Banach lattices. Then $T : E \to F$ is *positive* if $Tx \geq 0$ in F whenever $x \geq 0$ in E, and T is *regular* if $T = T_1 - T_2$, where T_1 and T_2 are positive. Such maps are necessarily continuous. Denote the space of these maps by $\mathcal{B}_r(E, F)$.

THEOREM 4.15. *Let E and F be Banach lattices. For $T \in \mathcal{B}_r(E, F)$, set*

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{B}(E, F)^+,\ |Tx| \leq S\,|x| \quad (x \in E^+)\}\,.$$

Then $(\mathcal{B}_r(E, F), \|\cdot\|_r)$ is a Banach space, and

$$\|T\|_r \geq \|T\| \quad (T \in \mathcal{B}_r(E, F))\,.$$

Further, $(\mathcal{B}_r(E), \|\cdot\|_r)$ is a unital Banach algebra. □

It is puzzling that this Banach algebra seems to have been very little studied; for example, it is not mentioned in [**1**].

THEOREM 4.16. *Let E and F be Banach lattices, and suppose that E is monotonically bounded and that F is Dedekind complete. Let $T \in \mathcal{B}(E, F)$. Then T is multi-bounded (with respect to the lattice multi-norms) if and only if T is regular, and so*

$$\mathcal{M}(E, F) = \mathcal{B}_r(E, F)\,.$$

Further,

$$\|T\|_{mb} = \|T\|_r = \|\,|T|\,\| = \|\,|T|\,\|_{mb} \quad (T \in \mathcal{M}(E, F))\,,$$

and

$$\|(T_1, \dots, T_n)\|_{mb,n} = \|\,|T_1| \vee \cdots \vee |T_n|\,\| = \|\,|T_1| \vee \cdots \vee |T_n|\,\|_{mb}$$

for $T_1, \dots, T_n \in \mathcal{M}(E, F)$ and each $n \in \mathbb{N}$.

COROLLARY 4.17. *Take p, q with $p, q \geq 1$, set $E = \ell^p$ and $F = \ell^q$, and regard $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ as multi-normed spaces with the standard (p, p)-multi-norm and (q, q)-multi-norm, respectively. Let $T \in \mathcal{B}(E, F)$. Then $T \in \mathcal{M}(E, F)$ if and only if $T \in \mathcal{B}_r(E, F)$, and, in this case, $\|T\|_{mb} = \|T\|_r$. This applies in particular when $p = q = 2$ to give the multi-bounded operators on a Hilbert space with the standard $(2, 2)$-multi-norm as the regular operators.* □

5. Lecture - The question of the dual

5.1. The problem. We wish to find a good definition of the 'dual' of a multi-normed space (again with our eyes on a standard course in functional analysis). This seems to be somewhat non-obvious.

A 'test question' for our future approach is the following.

Let $E = L^p(\Omega)$, where Ω is a measure space and $p > 1$, and let $\{E^n : n \in \mathbb{N}\}$ have the standard (p,q)-multi-norm, where $q \geq p$. Let p' and q' be the conjugate indices to p and q, respectively, and set $F = E' = L^{p'}(\Omega)$. Then we expect that the 'multi-dual' of the family $\{E^n : n \in \mathbb{N}\}$ will be $\{F^n : n \in \mathbb{N}\}$, with the standard (p',q')-multi-norm, and hence that $\{E^n : n \in \mathbb{N}\}$ is 'multi-reflexive'. Note that the 'standard (p',q')-multi-norm' only makes sense if $q' \geq p'$, and so this suggests that there will be no multi-dual when $q > p$, but that we might hope that the multi-dual of $\{E^n : n \in \mathbb{N}\}$ with the standard (p,p)-multi-norm is $\{F^n : n \in \mathbb{N}\}$, with the standard (p',p')-multi-norm.

We also expect that the 'multi-dual' of the lattice multi-norm on the family $\{E^n : n \in \mathbb{N}\}$, where E is a Banach lattice, will be the lattice multi-norm on $\{(E')^n : n \in \mathbb{N}\}$ (perhaps with some mild conditions on the lattice structure).

It is tempting to regard $\mathcal{M}(E,\mathbb{C})$ as the 'multi-dual' of a multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$. However recall that $\mathcal{M}(E,\mathbb{C}) = E'$ when we regard $\mathbb{C}$ as having its unique multi-norm structure, and that, as a multi-normed space, $\mathcal{M}(E,\mathbb{C})$ just has the minimum multi-norm; thus the approach of using this multi-normed space as a 'dual' is not satisfactory.

A second temptation is to start with the above multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and to look at the family $(((E')^n, \|\cdot\|_n') : n \in \mathbb{N})$. But this is an even worse failure: $(\|\cdot\|_n' : n \in \mathbb{N})$ is a **dual** multi-norm, not a multi-norm, on $\{(E')^n : n \in \mathbb{N}\}$.

Our solution to this question is to proceed through the notions of various decompositions of normed and multi-normed spaces.

5.2. Decompositions.

DEFINITION 5.1. *Let $(E, \|\cdot\|)$ be a normed space, and let*

$$E = E_1 \oplus \cdots \oplus E_k$$

be a direct sum decomposition of E. Then the decomposition is valid *if*

$$\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| \leq \|x_1 + \cdots + x_k\|$$

whenever $\zeta_1, \ldots, \zeta_k \in \mathbb{C}$ with $\max |\zeta_i| = 1$ and $x_1 \in E_1, \ldots, x_k \in E_k$.

Definition 5.2. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, and let* $E = E_1 \oplus \cdots \oplus E_k$ *be a direct sum decomposition of* E.

(i) *The decomposition is* small *if*

$$\|P_1 x_1 + \cdots + P_k x_k\| \leq \|(x_1, \ldots, x_k)\|_k \quad (x_1, \ldots, x_k \in E)\,.$$

(ii) *The decomposition is* orthogonal *if*

$$\|(x_1, \ldots, x_k)\|_k = \|x_1 + \cdots + x_k\| \quad (x_1 \in E_1, \ldots, x_k \in E_k)\,.$$

(Actually (ii) should be a little more complicated; see [**2**].)

Each valid decomposition is small with respect to the maximum multi-norm. Both small and orthogonal decompositions are valid. It is easy to find a small decomposition that is not orthogonal. I struggled to find an orthogonal decomposition that is not small - I believe that such an example exists; if so, it will be added to [**2**].

Example 5.3. Let H be a Hilbert space. Then the following are equivalent:

(a) $\{H_1, \ldots, H_k\}$ is an orthogonal decomposition of H with respect to the Hilbert multi-norm;

(b) $\{H_1, \ldots, H_k\}$ is a small decomposition of H;

(c) $\{H_1, \ldots, H_k\}$ is a valid decomposition of H;

(d) $\{H_1, \ldots, H_k\}$ is orthogonal in the classical sense that

$$H = H_1 \perp \cdots \perp H_k\,.$$

□

Example 5.4. Let Ω be a non-empty, compact space, and consider the multi-Banach space $((C(\Omega)^n, \|\cdot\|_n) : n \in \mathbb{N})$, where $(\|\cdot\|_n : n \in \mathbb{N})$ is the lattice multi-norm. Let $n \in \mathbb{N}$. Then $\{E_1, \ldots, E_n\}$ is an orthogonal decomposition of $C(\Omega)$ with respect to this multi-norm if and only if $E_i = C(\Omega_i)$ $(i = 1, \ldots, n)$, where $\{\Omega_1, \ldots, \Omega_n\}$ is a partition of Ω into closed subspaces. □

Example 5.5. Take p, q with $1 \leq p \leq q < \infty$, and set $E = \ell^p$. Let $\{E^n : n \in \mathbb{N}\}$ have the standard (p,q)-multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$,

In the case where $q \neq p$, there are no non-trivial orthogonal decompositions of $E = \ell^p$, and, in the case where $q = p$, the only non-trivial orthogonal decompositions of E are

$$\ell^p = \ell^p(S_1) \oplus \cdots \oplus \ell^p(S_k)\,,$$

where $\{S_1, \ldots, S_k\}$ is a partition of $\mathbb{N}$, and hence, regarding ℓ^p as a Banach lattice, we have

$$\ell^p = \ell^p(S_1) \perp \cdots \perp \ell^p(S_k)\,.$$

A similar remark applies to the complex (but not to the real) spaces $L^p(\Omega)$ by a clever argument of Hung Le Pham. $\square$

Question For a (complex) Banach lattice, is each orthogonal decomposition of E with respect to the lattice multi-norm already a classically orthogonal decomposition?

5.3. Families of decompositions.

DEFINITION 5.6. *Let $(E, \|\cdot\|)$ be a normed space, and consider a family $\mathcal{K} = \{(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) : \alpha \in A\}$, where A is an index set, $n_\alpha \in \mathbb{N}$ $(\alpha \in A)$, and*

$$E = E_{1,\alpha} \oplus \cdots \oplus E_{n_\alpha,\alpha}$$

is a direct sum decomposition of E for each $\alpha \in A$. The family $\mathcal{K}$ is closed *provided that the following conditions are satisfied:*

(C1) $(E_{\sigma(1),\alpha}, \dots, E_{\sigma(n_\alpha),\alpha}) \in \mathcal{K}$ *when* $(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) \in \mathcal{K}$ *and* $\sigma \in \mathfrak{S}_{n_\alpha}$*;*

(C2) $(E_{1,\alpha} \oplus E_{2,\alpha}, E_{3,\alpha}, \dots, E_{n_\alpha,\alpha}) \in \mathcal{K}$ *when* $(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) \in \mathcal{K}$ *and* $n_\alpha \geq 2$*;*

(C3) $\mathcal{K}$ *contains all trivial direct sum decompositions.*

The families of all direct sum decompositions, of all valid decompositions, of all small decompositions, and of all orthogonal decompositions are closed families of decompositions.

DEFINITION 5.7. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $\mathcal{K} = \{(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) : \alpha \in A\}$ be a closed family of orthogonal decompositions of E. Then the multi-normed space is* orthogonal with respect to $\mathcal{K}$ *if*

$$\|(x_1, \dots, x_n)\|_n = \sup_{\alpha \in A} \{\|(P_{1,\alpha}x_1, \dots, P_{n_\alpha,\alpha}x_n)\|_n : n_\alpha = n\}$$

for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, where $P_{j,\alpha}$ is the projection onto $E_{j,\alpha}$.

In the case where the multi-normed space is orthogonal with respect to the above family $\mathcal{K}$, it follows that we have

$$\|(x_1, \dots, x_n)\|_n = \sup_{\alpha \in A} \{\|P_{1,\alpha}x_1 + \cdots + P_{n_\alpha,\alpha}x_n\| : n_\alpha = n\}$$

for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$.

The above family of multi-norms $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ specified by a closed family $\mathcal{K}$ of valid decompositions of E is denoted by

$$(\|\cdot\|_{n,\mathcal{K}} : n \in \mathbb{N})\,.$$

This is the multi-norm generated by $\mathcal{K}$.

Query: What are the conditions on a multi-norm that ensure that it is orthogonal with respect to **some** closed family of valid decompositions?

5.4. Examples of families of decompositions.

EXAMPLE 5.8. Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be the family of all trivial orthogonal decompositions of E. Then the family $\mathcal{K}$ is closed, and $\mathcal{K}$ generates the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$. The multi-normed space is orthogonal with respect to $\mathcal{K}$. $\square$

THEOREM 5.9. *Let E be a Dedekind complete Banach lattice, and let the family $\{E^n : n \in \mathbb{N}\}$ have the lattice multi-norm*

$$(\|\cdot\|_n : n \in \mathbb{N}) .$$

Then the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is orthogonal with respect to the family of all classically orthogonal decompositions of E. Thus the lattice multi-norm is the multi-norm generated by the family of all classically orthogonal decompositions of E. $\square$

5.5. The multi-dual space. Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of valid decompositions of E. Then $\mathcal{K}$ generates a multi-norm $(\|\cdot\|_{n,\mathcal{K}} : n \in \mathbb{N})$ on $\{E^n : n \in \mathbb{N}\}$.

DEFINITION 5.10. *Let $(E, \|\cdot\|)$ be a normed space, and let*

$$\mathcal{K} = \{(E_{1,\alpha}, \ldots, E_{n_\alpha,\alpha}) : \alpha \in A\}$$

be a closed family of valid decompositions of E. The dual *to the family $\mathcal{K}$ is*

$$\mathcal{K}' = \{(E'_{1,\alpha}, \ldots, E'_{n_\alpha,\alpha}) : \alpha \in A\} .$$

The multi-norm on $\{(E')^n : n \in \mathbb{N}\}$ generated by $\mathcal{K}'$ is denoted by

$$(\|\cdot\|^\dagger_{n,\mathcal{K}} : n \in \mathbb{N}) .$$

The multi-normed space $(((E')^n, \|\cdot\|^\dagger_{n,\mathcal{K}}) : n \in \mathbb{N})$ is the multi-dual *space.*

Thus we have the following **method to find a multi-dual:** Start with a multi-normed space; find a closed family that generates it (this is not always possible); if there is such a family, take the dual of this family; let this new family generate a multi-norm on the family of dual spaces. (There is a question of uniqueness because this dual multi-norm may depend on the family that generates the original multi-norm. However, a mild condition given in [**2**] ensures that the multi-dual structure depends only on the original multi-norm, and not on the family that generates it.)

EXAMPLE 5.11. Take $p \geq 1$, and let $E = \ell^p$, with the standard (p,p)-multi-norm. Then the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is orthogonal with respect to the closed family $\mathcal{K}$ of all classically orthogonal decompositions of E.

Suppose that $p > 1$, and denote the conjugate index to p by q; set $F = \ell^q$. Clearly the multi-dual space of $((E^n, \|\cdot\|_n^{(p,p)}) : n \in \mathbb{N})$ is the multi-Banach space $((F^n, \|\cdot\|_n^{(q,q)}) : n \in \mathbb{N})$, where $(\|\cdot\|_n^{(q,q)} : n \in \mathbb{N})$ is the standard (q,q)-multi-norm.

Suppose that $p = 1$, and set $F = \ell^\infty$. Clearly the multi-dual space of $((E^n, \|\cdot\|_n^{(1,1)}) : n \in \mathbb{N})$ is the multi-Banach space $((F^n, \|\cdot\|_n^{\min}) : n \in \mathbb{N})$. $\square$

EXAMPLE 5.12. Let H be a Hilbert space. Then the multi-normed space $((H^n, \|\cdot\|_n^H) : n \in \mathbb{N})$ is orthogonal with respect to the family of all orthogonal decompositions of H. It is easy to see that the multi-dual space of $((H^n, \|\cdot\|_n^H) : n \in \mathbb{N})$ is equal to itself. $\square$

EXAMPLE 5.13. Let E be a Dedekind complete Banach lattice, so that E' is also a Dedekind complete Banach lattice. Then the lattice multi-norms on $\{E^n : n \in \mathbb{N}\}$ and $\{(E')^n : n \in \mathbb{N}\}$ are generated by the families of all classically orthogonal decompositions of E and E', respectively.

Let $\mathcal{K}$ be the family of all classically orthogonal decompositions of E. Then each member of $\mathcal{K}'$ is an orthogonal decomposition of E', but there could be more orthogonal decompositions of E' than are given by members of $\mathcal{K}'$. When does the family $\mathcal{K}'$ generate the lattice multi-norm on E'?

Now suppose that the norm on E is order-continuous (which implies that E is Dedekind complete). Then the family $\mathcal{K}'$ does generate the lattice multi-norm on E' Does this occur more generally? $\square$

5.6. Second dual spaces. The following result can be regarded as a multi-normed form of the Hahn–Banach theorem.

THEOREM 5.14. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $\mathcal{K}$ be a closed family of orthogonal decompositions of E, and let $(\|\cdot\|_{n,\mathcal{K}}^{\dagger\dagger} : n \in \mathbb{N})$ be the multi-norm on $\{(E'')^n : n \in \mathbb{N}\}$ generated by $\mathcal{K}''$. Then the canonical embedding of E into E'' gives a multi-isometry if and only if the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is orthogonal with respect to the family $\mathcal{K}$.* $\square$

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